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SOLIDIFICATION OF A PLANE INGOT IN A WEDGE-LIKE MOLD WITH A FEEDER HEAD

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The nonstationary problem of the solidification of metal in a plane wedge-like mold with a feeder head has been solved by the variational method using the concept of the local thermodynamic potential, and the positions of the solidification front at any instants of time have been determined.

Keywords: *feeder head, mold, solidification front, functional, variation, stationary equation, nonstationary, liquid phase, solid phase, instants of time.*

Introduction. Mold tops have been used in metallurgy for a long time. They are intended for removing a shrinkage hole and harmful admixtures into a feeder head, thus improving the quality of cast metal in a casting. But the joint solution of the heat conduction equations written for a mold and feeder head, with the motion of solidification fronts being determined in them, has been made only by numerical methods [1, 2].

Statement of the Problem. In the present work an analytical solution of the problem of plane ingot solidification in a wedge-like mold with a feeder head whose cross section is in the form of two truncated prisms is suggested. In Fig. 1 the outer contour of the cross section of the mold cut by a vertical plane perpendicular to the side faces is shown by a bold line. After the mold is filled with metal, solidification begins mainly from the side outer wall. To solve the problem, we will isolate five regions: 1, 3) regions of liquid metal in the mold and feeder head; 2, 4) regions of solidified metal in the mold and feeder head; 5) region of the joining of the solutions of heat conduction equations written for the mold and feeder head.

We consider the problem in a cylindrical coordinate system R, φ in the mold and r, ξ in the feeder head. The system of heat conduction equations is written with neglect of the viscous friction of liquid metal and of the transverse velocity components v_φ and v_ξ in comparison with the longitudinal ones v_R and v_r. A two-dimensional case, where the coordinate $Z >> r$, is considered, and therefore the dependence on Z is not taken into account [3]. The system of heat conduction equations written for four regions has the form

$$\frac{\partial T_1}{\partial t} + v_R \frac{\partial T_1}{\partial R} = a_1 \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T_1}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 T_1}{\partial \varphi^2}\right], \quad R_{\text{fr}} \le R \le R_2, \quad 0 \le \varphi \le \varphi_{\text{fr}}; \tag{1}$$

$$\frac{\partial T_2}{\partial t} = a_2 \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T_2}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 T_2}{\partial \varphi^2}\right], \quad R_1 \le R \le R_{\text{fr}}, \quad \varphi_{\text{fr}} \le \varphi \le \alpha; \tag{2}$$

$$\frac{\partial T_3}{\partial t} + v_r \frac{\partial T_3}{\partial r} = a_1 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_3}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_3}{\partial \xi^2}\right], \quad r_2 \le r \le r_{\text{fr}}, \quad 0 \le \xi \le \xi_{\text{fr}}; \tag{3}$$

$$\frac{\partial T_4}{\partial t} = a_2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_4}{\partial R}\right) + \frac{1}{r^2}\frac{\partial^2 T_4}{\partial \xi^2}\right], \quad r_1 \le r \le r_{\text{fr}}, \quad \xi_{\text{fr}} \le \xi \le \beta. \tag{4}$$

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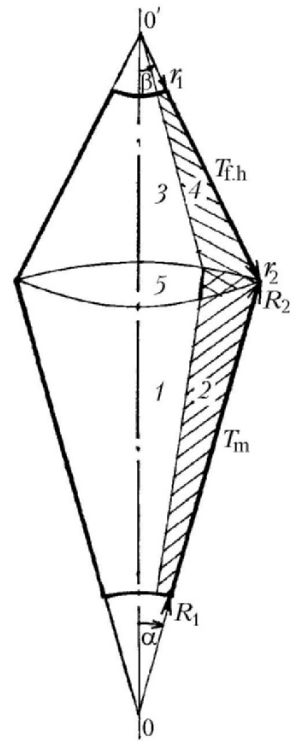

Fig. 1. Cross section of the mold and feeder head.

The system of equations (1)–(4) is solved together with the boundary conditions

$$T_1 = T_{cr} \text{ at } R = R_{fr}, \ T_1 = T_{cr} \text{ at } \varphi = \varphi_{fr}; \ T_1 = T_{13} \text{ at } R = R_2, \ \frac{\partial T_1}{\partial \varphi} = 0 \text{ at } \varphi = 0; \tag{5}$$

$$T_2 = T_m \text{ at } R = R_1, \ T_2 = T_m \text{ at } \varphi = \alpha; \ T_2 = T_{24} \text{ at } R = R_2, \ T_2 = T_{cr} \text{ at } R = R_{fr} \text{ and } \varphi = \varphi_{fr}; \tag{6}$$

$$T_3 = T_{13} \text{ at } r = r_2, \ T_3 = T_{cr} \text{ at } \xi = \xi_{fr}; \ T_3 = T_{cr} \text{ at } r = r_{fr}, \ \frac{\partial T_3}{\partial \xi} = 0 \text{ at } \xi = 0; \tag{7}$$

$$T_4 = T_{24} \text{ at } r = r_2, \ T_4 = T_{f.h} \text{ at } \xi = \beta, \ T_4 = T_{f.h} \text{ at } r = r_1, \ T_4 = T_{cr} \text{ at } \xi = \xi_{fr} \text{ and } r = r_{fr}. \tag{8}$$

The solidified metal thickness in the mold and feeder head is respectively equal to

$$\varepsilon_2 = R_{fr}(\alpha - \varphi_{fr}), \ \varepsilon_4 = r_{fr}(\beta - \xi_{fr}). \tag{9}$$

The conditions at the crystallization front are

$$\lambda_1 \left(\frac{1}{R} \frac{\partial T_1}{\partial \varphi} \right)_{\substack{\varphi=\varphi_{fr} \\ R=R_{fr}}} + L_1 \rho_1 \frac{\partial \varepsilon_2}{\partial t} = \lambda_2 \left(\frac{1}{R} \frac{\partial T_2}{\partial \varphi} \right)_{\substack{\varphi=\varphi_{fr} \\ R=R_{fr}}}, \tag{10}$$

$$\lambda_1 \left(\frac{1}{r} \frac{\partial T_3}{\partial \xi} \right)_{\substack{\xi=\xi_{fr} \\ r=r_{fr}}} + L_1 \rho_1 \frac{\partial \varepsilon_4}{\partial t} = \lambda_2 \left(\frac{1}{r} \frac{\partial T_4}{\partial \xi} \right)_{\substack{\xi=\xi_{fr} \\ r=r_{fr}}}. \tag{11}$$

From Eqs. (1)–(4) and boundary conditions (5)–(8) it is necessary to find four unknown functions for the temperatures, $T_1(R, \varphi, t)$, $T_2(R, \varphi, t)$, $T_3(r, \xi, t)$, and $T_4(r, \xi, t)$, which after substitution into (10) and (11) yield the laws of motion of solidification fronts in the mold and feeder head, whereas the joining of these fronts in region 5 makes it possible to determine the general character of the solidification front motion in a mold with a feeder head at any moment.

Solution of the Problem. We neglect metal solidification at the moment of pouring, since, first, the time of pouring is much smaller than the solidification time and, second. metal is usually poured superheated into a mold. Therefore at time $t = 0$ the solid phase is absent and

$$T_1 (R, \varphi, 0) = T_3 (r, \xi, 0) = T_{\text{in}} \quad \text{at} \quad R_2 > R > R_1 , \quad \alpha > \varphi > 0 , \quad r_2 > r > r_1 , \quad \beta > \xi > 0 . \tag{12}$$

At $\varphi = \alpha$, $R = R_1$, $\xi = \beta$, $r = r_1$, and $t > 0$ we have

$$T_1 (R_{\text{fr}}, \varphi_{\text{fr}}, t_{\text{fr}}) = T_{\text{cr}} , \quad T_3 (r_{\text{fr}}, \xi_{\text{fr}}, t_{\text{fr}}) = T_{\text{cr}} , \quad T_2 (R, \alpha, t) = T_{\text{m}} , \quad T_4 (r, \beta, t) = T_{\text{f.h}} . \tag{13}$$

The solution of Eq. (1) for R will be sought in the form of a linear function that satisfies boundary conditions (5): $T_1 = T_{\text{cr}}$ at $R = R_{\text{fr}}$, $T_1 = T_{13}$ at $R = R_2$, then

$$T_1 = T_{13} - \frac{R_2 - R}{R_2 - R_{\text{fr}}} (T_{13} - T_{\text{cr}}) . \tag{14}$$

The dependence on φ will be found from the solution of Eq. (1), written for a stationary case in the form

$$\frac{v_R}{a_1} R T_R - T_R - R T_{RR} - \frac{1}{R} T_{\varphi\varphi} = 0 , \tag{15}$$

where $T_R = \partial T/\partial R$; $T_{RR} = \partial^2 T/\partial R^2$; $T_{\varphi\varphi} = \partial^2 T/\partial \varphi^2$. The functional that corresponds to Eq. (15) will be written in the form

$$L = \int_{R_{\text{fr}}}^{R_2} \int_0^{\varphi_{\text{fr}}} \left[2 \frac{v_R}{a_1} R T_R^0 T + R T_R^2 + \frac{1}{R} T_\varphi^2 \right] dR d\varphi , \tag{16}$$

where $T_R^0 = \partial T^0/\partial R$ is the unvariable function of temperature.

We seek the function that minimizes the functional (16) in the form of the product of the function of R (14) and of the unknown function of φ:

$$T = T(r) f(\varphi) = \left[T_{13} - \frac{R_2 - R}{R_2 - r_{\text{fr}}} (T_{13} - T_{\text{cr}}) \right] f(\varphi) . \tag{17}$$

Calculating derivatives with respect to R and φ of (17), we obtain

$$T_R = \frac{T_{13} - T_{\text{cr}}}{R_2 - R_{\text{fr}}} f(\varphi) , \quad T_\varphi = \left[T_{13} - \frac{R_2 - R}{R_2 - R_{\text{fr}}} (T_{13} - T_{\text{cr}}) \right] f'(\varphi) . \tag{18}$$

Substituting (18) and (17) into (16) and integrating over R, we have

$$\int_0^{\varphi_{\text{fr}}} \left\{ A f^0 (\varphi) f(\varphi) + B f^2 (\varphi) + C [f'(\varphi)]^2 \right\} d\varphi = L , \tag{19}$$

where

$$A=\frac{v_R\,(T_{13}-T_{\rm cr})}{3a_1}\,[R_2\,(2T_{13}+T_{\rm cr})+R_{\rm fr}\,(T_{13}+2T_{\rm cr})]\,;\quad B=\frac{R_2+r_{\rm fr}}{2\,(R_2-R_{\rm fr})}\,(T_{13}-T_{\rm cr})^2\,;$$

$$C=\frac{T_{13}-T_{\rm cr}}{2\,(R_2-R_{\rm fr})}\,[-R_{\rm fr}\,(3T_{13}+T_{\rm cr})+R_2\,(T_{13}+3T_{\rm cr})]+\frac{2\,(R_{\rm fr}T_{13}-R_2T_{\rm cr})^2}{(R_2-R_{\rm fr})^2}\ln\frac{R_2}{R_{\rm fr}}\,. \tag{20}$$

The vanishing of the variation of the functional (19) corresponds to the best choice of the function $f(\varphi)$:

$$\delta L=\frac{\partial L}{\partial f\,(\varphi)}-\frac{\partial}{\partial\,(\varphi)}\frac{\partial L}{\partial f'(\varphi)}=0\,. \tag{21}$$

We will find derivatives of the integrand in (19). This results in

$$\frac{\partial L}{\partial f\,(\varphi)}=Af^0\,(\varphi)+Bf\,(\varphi)\,,\quad \frac{\partial L}{\partial f'(\varphi)}=2Cf'(\varphi)\,;\quad \frac{\partial}{\partial\varphi}\frac{\partial L}{\partial f'(\varphi)}=2Cf''(\varphi)\,. \tag{22}$$

In the first derivative (22) the function $f^0(\varphi)$ is equated to $f(\varphi)$ after the completion of the process of variation. As a result of the transformation and subject to (21) and (22), we obtain

$$f''(\varphi)-\frac{A+2B}{2C}f\,(\varphi)=0\,.$$

The following function will be the solution of this equation [4]:

$$f\,(\varphi)=C_1\cosh K\varphi+C_2\sinh K\varphi\,, \tag{23}$$

where $K=\sqrt{(A+2B)/2C}$.

The constants C_1 and C_2 can be found from the boundary conditions for the function $f(\varphi)$: $T=T_{\rm cr}$ at $\varphi=\varphi_{\rm fr}$, $\partial T/\partial\varphi=0$ at $\varphi=0$. Taking them into account in (23) and calculating, we obtain $C_2=0$, $C_1=1/(\cosh K\varphi_{\rm fr})$. Substituting them into (23) and the latter equation into (17), we find the solution of Eq. (15):

$$T=\left[T_{13}-\frac{R_2-R}{R_2-R_{\rm fr}}\,(T_{13}-T_{\rm cr})\right]\frac{\cosh K\varphi}{\cosh K\varphi_{\rm fr}}\,. \tag{24}$$

Now we will find the solution of Eq. (1) for a nonstationary case. We will use the same method as for determining the dependence on φ. The functional corresponding to Eq. (1) will be written as

$$L=\int_0^{t_{\rm fr}}\int_0^{\varphi_{\rm fr}}\int_{R_{\rm fr}}^{R_2}\left(\frac{2v_R}{a_1}RT_R^0T+2\frac{R}{a_1}TT_t^0+RT_R^2+\frac{1}{R}T_\varphi^2\right)dRd\varphi dt\,, \tag{25}$$

Here T_R^0 and T_t^0 are the nonvariable functions of temperature; $t_{\rm fr}$ is the time corresponding to the coordinates $R_{\rm fr}$ and $\varphi_{\rm fr}$.

The solution of Eq.(1) is sought in the form

$$T=\left[T_{13}-\frac{R_2-R}{R_2-R_{\rm fr}}\,(T_{13}-T_{\rm cr})\right]\frac{\cosh K\varphi}{\cosh K\varphi_{\rm fr}}f\,(t)\,, \tag{26}$$

where $f(t)$ is the sought-for function of time.

We will find the derivatives with respect to R and φ of Eq. (26). As a result we have

$$T_R=\frac{T_{13}-T_{\rm cr}}{R_2-R_{\rm fr}}\frac{\cosh K\varphi}{\cosh K\varphi_{\rm fr}}f\,(t)\,,\quad T_\varphi=\left[T_{13}-\frac{R_2-R}{R_2-R_{\rm fr}}\,(T_{13}-T_{\rm cr})\right]\frac{K\sinh K\varphi}{\cosh K\varphi_{\rm fr}}f\,(t)\,,$$

$$T_t^0 = \left[T_{13} - \frac{R_2 - R}{R_2 - R_{\text{fr}}} (T_{13} - T_{\text{cr}}) \right] \frac{\cosh K\varphi}{\cosh K\varphi_{\text{fr}}} f_0'(t) , \quad (27)$$

where $f_0'(t)$ is a nonvariable function of time. Substituting (26) and (27) into (25) and integrating over R and φ, we obtain

$$L = \int_0^{t_{\text{fr}}} \left\{ Df^0(t) f(t) + Ef(t) f_0'(t) + Ff^2(t) \right\} dt , \quad (28)$$

where D, E, and F are the constants of integration over r and φ. We will find the equation for determining the function $f(t)$ by taking the variation of (28) and equating it to zero:

$$\frac{\partial L}{\partial f(t)} = (D + 2F) f(t) + Ef'(t) = 0 . \quad (29)$$

The function $f(t)$ satisfies Eq. (29) as an exponent in time:

$$f(t) = C \exp\left(-\frac{D + 2F}{E} t \right). \quad (30)$$

We will find the constant C from the condition at the crystallization front:

$$T = (R = R_{\text{fr}}, \varphi = \varphi_{\text{fr}}, t = t_{\text{fr}}) = T_{\text{cr}} . \quad (31)$$

For this purpose, we will write the solution of the nonstationary problem as a product of three functions of the variables R, φ, and t (26):

$$T_1 = \left[T_{13} - \frac{R_2 - R}{R_2 - R_{\text{fr}}} (T_{13} - T_{\text{cr}}) \right] \frac{\cosh K\varphi}{\cosh K\varphi_{\text{fr}}} C \exp\left(-\frac{D + 2F}{E} t \right). \quad (32)$$

Using (31), we find

$$C = \exp\left(\frac{D + 2F}{E} t_{\text{fr}} \right). \quad (33)$$

Substituting (33) into (32), we obtain the temperature distribution in the liquid phase:

$$T_1 = \left[T_{13} - \frac{R_2 - R}{R_2 - R_{\text{fr}}} (T_{13} - T_{\text{cr}}) \right] \frac{\cosh K\varphi}{\cosh K\varphi_{\text{fr}}} \exp\left(-\frac{D + 2F}{E} (t - t_{\text{fr}}) \right). \quad (34)$$

In the same way we will find the temperature distribution in the solid phase of the solidified metal (2). For this purpose, it is necessary to solve Eq. (2) with boundary conditions (6). In the stationary case, Eq. (2) will take the form

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T_2}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 T_2}{\partial \varphi^2} = 0 , \quad (35)$$

We will prescribe solution (35) for R in the form that satisfies boundary conditions (6):

$$T_2 = T_{\text{m}} \text{ at } R = R_1 , \quad T_2 = T_{\text{cr}} \text{ at } R = R_{\text{fr}} . \quad (36)$$

Then

$$T_2 = T_{cr} + \frac{R_{fr} - R}{R_{fr} - R_1} (T_m - T_{cr}) \,. \tag{37}$$

The dependence over φ will be found from the solution of Eq. (35). For this purpose, we will write the functional corresponding to this equation. Subscript 2 at the temperature will be omitted:

$$L = \int_{R_1}^{R_{fr}} \int_{\varphi_{fr}}^{\alpha} \left(RT_R^2 + \frac{1}{R} T_\varphi^2 \right) dR d\varphi \,. \tag{38}$$

The temperature as the function of two variables R and φ will be presented in the form

$$T = \left[T_{cr} + \frac{R_{fr} - R}{R_{fr} - R_1} (T_m - T_{cr}) \right] f(\varphi) \,. \tag{39}$$

Having determined the derivatives with respect to R and φ, we obtain

$$T_R = -\frac{T_m - T_{cr}}{R_{fr} - R_1} f(\varphi) \,, \quad T_\varphi = \left[T_{cr} + \frac{R_{fr} - R}{R_{fr} - R_1} (T_m - T_{cr}) \right] f'(\varphi) \,. \tag{40}$$

Substitution of (39) and (40) into (38) yields

$$L = \int_{R_1}^{R_{fr}} \int_{\varphi_{fr}}^{\alpha} \left\{ R \frac{(T_m - T_{cr})^2}{(R_{fr} - R_1)^2} f^2(\varphi) + \left[T_{cr} + \frac{R_{fr} - R}{R_{fr} - R_1} (T_m - T_{cr}) \right]^2 [f'(\varphi)]^2 \right\} dR d\varphi \,. \tag{41}$$

Integration over R gives

$$L = \int_{\varphi_{fr}}^{\alpha} \left\{ A_2 f^2(\varphi) + B_2 [f'(\varphi)]^2 \right\} d\varphi \,, \tag{42}$$

where

$$A_2 = \frac{R_{fr} + R_1}{2(R_{fr} - R_1)} (T_m - T_{cr})^2 \,;$$

$$B_2 = T_{cr}^2 \ln \frac{R_{fr}}{R_1} - 2T_{cr} (T_{cr} - T_m) \left(\frac{R_{fr}}{R_{fr} - R_1} \ln \frac{R_{fr}}{R_1} - 1 \right) + (T_{cr} - T_m)^2 \left[\frac{R_{fr}}{(R_{fr} - R_1)^2} \ln \frac{R_{fr}}{R_1} - \frac{3R_{fr} - R_1}{2(R_{fr} - R_1)} \right] \tag{43}$$

Having calculated the variation of (42) and equated it to zero, we obtain

$$A_2 f(\varphi) - B_2 f''(\varphi) = 0 \,. \tag{44}$$

The following function will be the solution of Eq. (44):

$$f(\varphi) = C_1 \cosh \varphi K_2 + C_2 \sinh \varphi K_2 \,, \tag{45}$$

where $K_2 = \sqrt{A_2/B_2}$. The constants C_1 and C_2 will be found from the boundary conditions for the function T_2:

$$T_2 = T_{cr} \;\; \text{at} \;\; \varphi = \varphi_{fr} \,, \quad T_2 = T_m \;\; \text{at} \;\; \varphi = \alpha \,. \tag{46}$$

Using (45) and (46), from (39) we obtain

$$C_1 \cosh \alpha K_2 + C_2 \sinh \alpha K_2 = \frac{T_{\mathrm{m}}}{T_{\mathrm{cr}} + \dfrac{R_{\mathrm{fr}} - R}{R_{\mathrm{fr}} - R_1}(T_{\mathrm{m}} - T_{\mathrm{cr}})} . \tag{47}$$

In (47) the variables φ_{fr}, R_{fr}, and R will be replaced by constant average values $\alpha/2$, $(R_1 + R_2)/2$, $(R_1 + R_2)/2$. Then we obtain

$$C_1 \cosh \frac{\alpha}{2} K_2 + C_2 \sinh \frac{\alpha}{2} K_2 = 1 , \quad C_1 \cosh \alpha K_2 + C_2 \sinh \alpha K_2 = \frac{T_{\mathrm{m}}}{T_{\mathrm{cr}}} . \tag{48}$$

Solving the system of equations (48), we find

$$C_1 = 2 \coth \frac{\alpha}{2} K_2 - \frac{T_{\mathrm{m}}}{T_{\mathrm{cr}}} , \quad C_2 = \frac{T_{\mathrm{m}}}{T_{\mathrm{cr}}} \coth \frac{\alpha}{2} K_2 - \frac{\cosh \alpha K_2}{\sinh \dfrac{\alpha}{2} K_2} . \tag{49}$$

Equation (45), subject to (49), yields

$$f(\varphi) = \left(2 \cosh \frac{\alpha}{2} K_2 - \frac{T_{\mathrm{m}}}{T_{\mathrm{cr}}} \right) \cosh \varphi K_2 + \left(\frac{\dfrac{T_{\mathrm{m}}}{T_{\mathrm{cr}}} \cosh \dfrac{\alpha}{2} K_2 - \cosh \alpha K_2}{\sinh \dfrac{\alpha}{2} K_2} \right) \sinh \varphi K_2 . \tag{50}$$

We will find the solution of Eq. (2) with allowance for the nonstationarity of the process. The functional corresponding to this equation has the form

$$L = \int_0^{t_{\mathrm{fr}}} \int_{\varphi_{\mathrm{fr}}}^{\alpha} \int_{R_1}^{R_{\mathrm{fr}}} \left[2 \frac{R}{a_2} T T_t^0 + R T_R^2 + \frac{1}{R} T_\varphi^2 \right] dR d\varphi dt , \tag{51}$$

where t_{fr} is the time corresponding to the coordinates R_{fr} and φ_{fr}. We seek the solution of Eq. (51) in the form

$$T_2 = \left[T_{\mathrm{cr}} + \frac{R_{\mathrm{fr}} - R}{R_{\mathrm{fr}} - R_1} (T_{\mathrm{m}} - T_{\mathrm{cr}}) \right] \left[C_1 \cosh \varphi K_2 + C_2 \sinh \varphi K_2 \right] f(t) , \tag{52}$$

where $f(t)$ is the function sought; C_1 and C_2 are defined by Eqs. (49).

Having omitted the subscript 2 at T in (52) and calculated the derivatives T_R, T_φ, and T_t, we obtain

$$T_R = \frac{T_{\mathrm{cr}} - T_{\mathrm{m}}}{R_{\mathrm{fr}} - R_1} \left(C_1 \cosh \varphi K_2 + C_2 \sinh \varphi K_2 \right) f(t) ,$$

$$T_\varphi = \left[T_{\mathrm{cr}} + \frac{R_{\mathrm{fr}} - R}{R_{\mathrm{fr}} - R_1} (T_{\mathrm{m}} - T_{\mathrm{cr}}) \right] \left[C_1 K_2 \sinh \varphi K_2 + C_2 K_2 \cosh \varphi K_2 \right] f(t) , \tag{53}$$

$$T_t = \left[T_{\mathrm{cr}} + \frac{R_{\mathrm{fr}} - R}{R_{\mathrm{fr}} - R_1} (T_{\mathrm{m}} - T_{\mathrm{cr}}) \right] \left[C_1 \cosh \varphi K_2 + C_2 \sinh \varphi K_2 \right] f'(t) .$$

Substituting (53) into (51) and integrating over R and φ, we obtain

$$L = \int_0^{t_{fr}} \left[M_2 f(\varphi) f_0'(\varphi) + N_2 f^2(\varphi) + P_2 f^2(\varphi) \right] dt \,, \tag{54}$$

where M_2, N_2, and P_2 are the constants of integration over R and φ. The following equation yields the variation of (54) over $f(t)$:

$$f'(t) + 2\frac{N_2 + P_2}{M_2} f(t) = 0 \,.$$

Its solution is the exponential function of time

$$f(t) = C \exp\left(-2\frac{N_2 + P_2}{M_2} t \right). \tag{55}$$

The constant C will be determined from the condition at the crystallization front:

$$T_2 = T_{cr} \,, \quad R = R_{fr} \,, \quad \varphi = \varphi_{fr} \,, \quad t = t_{fr} \,. \tag{56}$$

For this purpose, with (55) taken into account and with the use of conditions (56), from Eq. (52) we obtain

$$C = \frac{\exp\left(2\frac{N_2 + P_2}{M_2} t_{fr} \right)}{C_1 \cosh \varphi_{fr} K_2 + C_2 \sinh \varphi_{fr} K_2} \,. \tag{57}$$

Subject to (57), Eq. (52) takes the form

$$T_2 = \left[T_{cr} + \frac{R_{fr} - R}{R_{fr} - R_1} (T_m - T_{cr}) \right] \frac{C_1 \cosh \varphi K_2 + C_2 \sinh \varphi K_2}{C_1 \cosh \varphi_{fr} K_2 + C_2 \sinh \varphi_{fr} K_2} \exp\left(-2\frac{N_2 + P_2}{M_2} (t - t_{fr}) \right). \tag{58}$$

Equation (58) describes the dependence of the temperature in the solid phase on the variables R, φ, and t.

Knowing the nonstationary distribution of temperature in the liquid metal (Fig. 1, region 1) — Eq. (34), as well as the distribution of temperature in the solid phase (region 2) — Eq. (58), we determine the crystallization front motion in the mold. For this purpose, we rewrite the condition at the crystallization front (10) in the form

$$v_{cr1} = \frac{\partial \varepsilon_2}{\partial t} = \frac{1}{L_1 \rho_1} \left[\lambda_2 \left(\frac{1}{R} \frac{\partial T_2}{\partial \varphi} \right)_{\substack{R = R_{fr} \\ \varphi = \varphi_{fr}}} - \lambda_1 \left(\frac{1}{R} \frac{\partial T_1}{\partial \varphi} \right)_{\substack{R = R_{fr} \\ \varphi = \varphi_{fr}}} \right]. \tag{59}$$

We will find the derivatives $\partial T_1 / \partial \varphi$ and $\partial T_2 / \partial \varphi$ of expression (34) and (58). Taking into account the conditions at the crystallization front $R = R_{fr}$, $\varphi = \varphi_{cr}$, and $t = t_{fr}$, we obtain

$$\frac{\partial T_1}{\partial \varphi} = T_{cr} \frac{K \sinh K\varphi_{fr}}{\cosh K\varphi_{fr}} \,; \qquad \frac{\partial T_2}{\partial \varphi} = T_{cr} \frac{C_1 K_2 \sinh \varphi_{fr} K_2 + C_2 K_2 \cosh \varphi_{fr} K_2}{C_1 \cosh \varphi_{fr} K_2 + C_2 \sinh \varphi_{fr} K_2} \,. \tag{60}$$

Differentiating ε_2 with respect to time at a constant value of φ_{fr}, from Eq. (9) we obtain

$$\frac{d\varepsilon_2}{dt} = (\alpha - \varphi_{fr}) \frac{dR_{fr}}{dt} \,. \tag{61}$$

From Eq. (59), subject to (60) and (61), having multiplied by R_{fr}, we find

$$R_{fr}\frac{dR_{fr}}{dt}(\alpha-\varphi_{fr})=\frac{T_{cr}}{L_1\rho_1}\left[\lambda_2 K_2\frac{\left(C_1\sinh\varphi_{fr}K_2+C_2\cosh\varphi_{fr}K_2\right)}{\left(C_1\cosh\varphi_{fr}K_2+C_2\sinh\varphi_{fr}K_2\right)}-\lambda_1\frac{K\sinh K\varphi_{fr}}{\cosh K\varphi_{fr}}\right], \tag{62}$$

where C_1, C_2, K_2, and K are defined by Eqs. (49), (43), (45), and (23). Integrating Eq. (62) over R_{fr} at a constant value of $(\alpha-\varphi_{fr})$, we obtain

$$R_{fr}=\sqrt{\frac{2T_{cr}C_*}{L_1\rho_1(\alpha-\varphi_{fr})}t+R_1^2}\,, \tag{63}$$

where C_* is the expression in square brackets in Eq. (62), and the constant R_1^2 accounts for the start of crystallization at t = 0 and R_{fr} = R_1. In the same way we find the solution of Eqs. (3) and (4) that describe the nonstationary distribution of temperature in the liquid and solid phases in the feeder head with boundary conditions (7) and (8):

$$T_3=\left[T_{13}-\frac{r_2-r}{r_2-r_{fr}}(T_{13}-T_{cr})\right]\frac{\cosh K_3\xi}{\cosh K_3\xi_{fr}}\exp\left(-\frac{D_3+2F_3}{E_3}(t-t_{fr})\right), \tag{64}$$

$$T_4=\left[T_{cr}+\frac{r_{fr}-r}{r_{fr}-r_1}(T_{f.h}-T_{cr})\right]\frac{C_6\cosh\xi K_4+C_7\sinh\xi K_4}{C_6\cosh\xi_{fr}K_4+C_7\sinh\xi_{fr}K_4}\exp\left(-2\frac{N_4+P_4}{M_4}(t-t_{fr})\right), \tag{65}$$

where

$$K_3=\sqrt{\frac{A_3+2B_3}{2C_3}}\;;\;\; A_3=-\frac{v_3}{3a_1}(T_{13}-T_{cr})\left[r_2(2T_{13}+T_{cr})+r_{fr}(T_{13}+2T_{cr})\right];\;\; B_3=\frac{(T_{13}-T_{cr})^2}{2(r_2-r_{cr})}(r_{cr}+r_2)\;;\;\; K_4\sqrt{\frac{A_4}{B_4}}$$

$$C_3=\frac{1}{2(r_2-r_{fr})^2}\left\{-(r_2-r_{fr})(T_{13}-T_{cr})\left[-r_{fr}(3T_{13}+T_{cr})+r_2(T_{13}+3T_{cr})\right]-2(r_{fr}T_{13}-r_2T_{cr})^2\ln\frac{r_2}{r_{fr}}\right\};$$

$$C_6=2\cosh\frac{\beta}{2}K_4-\frac{T_{f.h}}{T_{cr}}\;;\;\; C_7=\frac{T_{f.h}}{T_{cr}}\coth\frac{\beta}{2}K_4-\frac{\cosh\beta K_4}{\sinh\frac{\beta}{2}\sqrt{K_4}}\;;$$

$$A_4=\frac{r_{fr}+r_1}{2(r_{fr}-r_1)}(T_{cr}-T_{f.h})^2;\;\; B_4=T_{cr}^2\ln\frac{r_{fr}}{r_1}-2T_{cr}(T_{cr}-T_{f.h})\left(\frac{r_{fr}}{r_{fr}-r_1}\ln\frac{r_{fr}}{r_1}-1\right)+(T_{cr}-T_{f.h})^2\left[\frac{r_{fr}^2}{(r_{fr}-r_1)^2}\ln\frac{r_{fr}}{r_1}-\frac{3r_{fr}-r_1}{2(r_{fr}-r_1)}\right].$$

Taking the derivatives with respect to ξ of (64) and (65) and substituting them into (11), after integration over r_{fr} at a constant value of $(\beta-\xi_{fr})$, we obtain

$$r_{fr}=\sqrt{\frac{2T_{cr}C_{**}}{L_1\rho_1(\beta-\xi_{fr})}t+r_1^2}\,, \tag{66}$$

$$C_{**}=\lambda_2\sqrt{\frac{A_4}{B_4}}\frac{(C_6\sinh\xi_{fr}\sqrt{A_4/B_4}+C_7\cosh\xi_{fr}\sqrt{A_4/B_4})}{(C_6\cosh\xi_{fr}\sqrt{A_4/B_4}+C_7\sinh\xi_{fr}\sqrt{A_4/B_4})}-\lambda_1K_3\tanh K_3\xi_{fr}\,.$$

Discussion of Results. Using Eqs. (63) and (66), we carried out numerical calculations of the solidification front position at definite moments at the following values of the geometric dimensions of the mold and feeder head

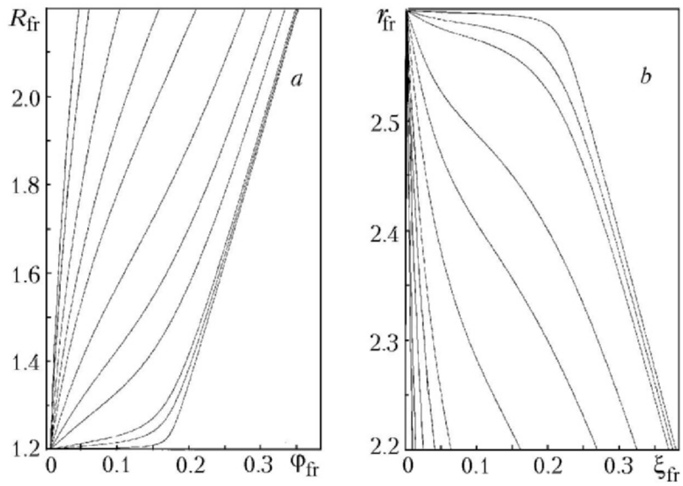

Fig. 2. Position of the solidification front in the mold (a) and feeder head (b) at the instants of time 1, 5, 10, 50, 100, 200, 400, 600, 10^3, $2{\cdot}10^3$, and $3{\cdot}10^3$ sec. Counting of the curves from right to left.

and at the parameters of liquid and solid steel: the initial temperature of the liquid metal poured into the mold T_{13} = 1873 K, the solidification temperature T_{cr} = 1733 K, the temperature of the mold walls T_m = 1433 K, the temperature of the feeder head walls $T_{f.h}$ = 1533 K, the lower radius of the mold R_1 = 1.2 m, the upper radius of the mold R_2 = 2.2 m, the upper radius of the feeder head r_1 = 0.5 m, the lower radius of the feeder head r_2 = 0.9 m, the thermal diffusivity of steel $a_1 = 4.5{\cdot}10^{-6}$ m^2/sec, the crystallization heat $L_1 = 2.72{\cdot}10^5$ J/kg, the steel density $\rho = 7.31{\cdot}10^3$ kg/m^3, the thermal conductivity of the liquid phase λ_1 = 26.5 W/(m·K), the thermal conductivity of the solid phase λ_2 = 30.3 W/(m·K), the velocity of convection in the mold $v_R = 10^{-2}$ m/sec, the velocity of convection in the feeder head $v_r = 10^{-2}$ m/sec, the conicity angle of the mold $\alpha = 10^{\circ}$, and the conicity angle of the feeder head $\beta = 25^{\circ}$.

The position of the solidification fronts in the mold and feeder head was calculated at the instants of time 1, 5, 10, 50, 100, 200, 400, 600, 10^3, $2{\cdot}10^3$, and $3{\cdot}10^3$ sec. From the equations and graphs it is seen that the solidification of the metal follows the law of the square root, and the solidification front moves most rapidly from the mold corners (Fig. 2a) and feeder head (Fig. 2b). It should be noted that the solidification rate in the feeder head is higher than in the mold at the given values of temperatures $T_{f.h}$ and T_m, which generally is undesirable, since in this case a sink hole is formed inside the mold. Therefore the feeder head walls should be kept at a higher temperature to slow down the solidification of the feeder head.

NOTATION

A, B, C, A_2, B_2, C_2, A_3, B_3, C_3, A_4, B_4, D_3, F_3, E_3, N_4, P_4, and M_4, constants of integration over R, r, and φ, ξ; a_1, a_2, thermal diffusivities of liquid and solid metal, m^2/sec; L_1, crystallization heat, J/kg; L, functional or Lagrangian; R_1, R_2 and r_1, r_2, lower and upper radii of the mold and feeder head, m; R, φ, cylindrical coordinates of points inside the mold; r, ξ, cylindrical coordinates of points inside the feeder head; T_1, T_3, temperatures of liquid metal the mold and feeder head, K; T_2, T_4, temperatures of solid metal inside the mold and feeder head, K; T_{cr}, crystallization temperature, K; T_m, $T_{f.h}$, temperatures of the bottom and side surface of the mold and feeder head, K; T_{in}, initial temperature of casting, K; t, time, sec; t_{fr}, time at the crystallization front, sec; T_R^0, T_t^0, unvariable derivatives of temperature over radius and time; T, temperature function; T_r, T_{rr}, T_φ, $T_{\varphi\varphi}$, and T_t, the first and second derivatives over radius, angle, and time; v_R, v_r, radial components of velocity in the mold and feeder head, m/sec; α, β, conicity angles of the side walls of the mold and feeder head, deg; ε_2, ε_4, thickness of solidified metal in the mold and feeder head, m; λ_1, λ_2, thermal conductivities of liquid and solid metal, W/(m·K); ρ_1, ρ_2, densities of liquid and solid metal, kg/m^3; ρ, average density of metal, kg/m^3; φ_{fr}, R_{fr}, ξ_{fr}, and r_{fr}, azimuthal and radial coordinated of the point at the solidification front in the mold and feeder head. Subscripts: cr, crystallization; f.h, feeder-head surface; fr, front; in, initial; m, mold surface.

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